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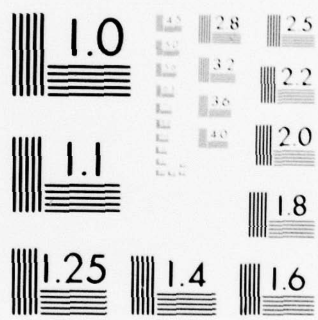

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ANALYSIS OF THICK RECTANGULAR PLATES LAMINATED OF BIMODULUS COMPOSITE MATERIALS

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ANALYSIS OF THICK RECTANGULAR PLATES LAMINATED OF BIMODULUS COMPOSITE MATERIALS

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A mixed finite-element analysis is presented for static behavior of rectangular plates having finite transverse shear moduli and different elastic properties depending upon whether or not the fiber-direction strains are tensile or compressive. As a benchmark to evaluate the validity and accuracy of the finite-element analysis, a closed-form solution is presented for the particular case of an unsymmetric-cross-ply plate having freely supported edges and subjected to a sinusoidally distributed normal-pressure loading.

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Index categories: Aircraft Structural Materials; Structural Composite Materials; Structural Static Analysis.

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Nomenclature

A, A_{ij} = stretching stiffnesses for transversely isotropic and cross-ply orthotropic plates

a, b = plate dimensions in x and y directions

B, B_{ij} = bending-stretching coupling stiffnesses for transversely isotropic and cross-ply orthotropic plates

C_{rs} = coefficients defined in Eqs. (15)

D, D_{ij} = bending stiffnesses for transversely isotropic and cross-ply orthotropic plates

d_x = $\partial(\ )/\partial x$

E_c, E_t = compressive and tensile Young's moduli for transversely isotropic bimodulus material

E_f, E_m = fiber and matrix Young's moduli

E_1, E_2, E_3 = Young's moduli in directions x,y,z

F_i^{α} = generalized force components defined in Eqs. (20)

G_{13}, G_{23} = longitudinal-thickness and transverse-thickness shear moduli of orthotropic material

G_{zc}, G_{zt} = thickness-shear moduli for transversely isotropic bimodulus material

h = total thickness of plate

K = shear-correction coefficient for transversely isotropic plate

K_4, K_5 = shear-correction coefficients for cross-ply orthotropic plate

K_{ij} = matrix coefficients defined in Eqs. (20)

L_{rs} = linear differential operators defined in Eqs. (8)

M_i, N_i = stress couples and inplane stress resultants

$\mathcal{N}_i$ = interpolation function at node i

n = number of nodes per element

Q_c, Q_t = compressive and tensile plane-stress-reduced stiffnesses for isotropic bimodulus material

Q_x, Q_y = thickness-shear stress resultants

$\bar{Q}, \Delta Q$ = $(1/2)(Q_c+Q_t)$, $Q_c - Q_t$

$Q_{ijk\ell}$ = plane-stress-reduced stiffnesses for orthotropic bimodulus material

q, q_o = normal pressure and its peak value

S, S_{ij} = thickness-shear stiffnesses for transversely isotropic and cross-ply orthotropic plates

$\mathcal{S}_{ij}$ = matrix coefficients defined in Eqs. (20)

U,V,W = midplane displacement coefficients (amplitudes of u^o, v^o, w^o)

u,v,w = displacements in x,y,z directions

u^o, v^o, w^o = midplane displacements in x,y,z directions

V_f, V_f^* = fiber volume fraction and effective fiber volume fraction

X,Y = bending-slope coefficients (amplitudes of ψ_x, ψ_y)

x,y,z = plate coordinates in longitudinal, transverse, and downward thickness directions

Z, Z_x, Z_y = z_n/h, z_{nx}/h, z_{ny}/h

z_n = neutral-surface position for isotropic square plate

z_{nx}, z_{ny} = neutral-surface positions associated with $\varepsilon_x=0$ and $\varepsilon_y=0$

α, β = π/a, π/b

$\varepsilon_j, \varepsilon_j^o$ = strain component at arbitrary location and at midplane

κ_j = curvature component

ν_f, ν_m = fiber and matrix Poisson's ratios

ν_{12}, ν_{23} = major (longitudinal-transverse) and transverse-thickness Poisson's ratios

σ_i = stress component

ϕ_e, ϕ_e^i = typical variable in general and its value at node i

ψ_x, ψ_y = bending slopes in xz and yz planes

$(\ )_{,x}$ = $\partial(\ )/\partial x$

Subscripts:

i,j = 1,2,6 contracted indices

k = 1(t or tension), 2(c or compression)

ℓ = layer number

Introduction

The increasing use of composite materials in structures has led to the requirement of more realistic mathematical modeling of the material behavior and incorporation of this more realistic model into structural analyses. It has been found that certain fiber-reinforced materials, especially those with very soft matrices (for example, cord-rubber composites), exhibit quite different elastic behavior depending upon whether the fiber-direction strain is tensile or compressive[1-3]. As a first approximation, the stress-strain behavior of such materials is often represented as being bilinear, with different slopes (elastic properties) depending upon the sign of the fiber-direction strain. Such a material is called a bimodulus composite material, and it has been shown that the fiber-governed symmetric-compliance model proposed in Ref. 4 agrees well with experimental data for several materials with drastically different elastic properties in tension and compression.

To the best of the present investigators' knowledge, the only previous analyses of plates laminated of bimodulus composite materials are all limited to thin plates. Jones and Morgan[5] considered cylindrical bending of a finite-width cross-ply strip; Kincannon et al.[6] considered cross-ply elliptic plates. Rectangular plates were treated by Bert et al.[7] using a closed-form solution and by Reddy[8] using mixed finite elements.

Apparently the only previous analyses involving thick plates of bimodulus material are those of Shapiro[9] using a stress-function elasticity approach for isotropic circular plates and of Kamiya[10] using an energy approach for cylindrical bending of finite-width isotropic strips.

Of course, for thick plates laminated of ordinary (not bimodulus) materials, there have been a number of analyses, such as those of Whitney[11], Whitney and Pagano[12], and Turvey[13], for example.

The analyses presented here are believed to be the very first analyses of thick plates that are finite in two directions and laminated of bimodulus composite materials.

Formulation

The basic theory of laminated anisotropic thick plates used by Whitney and Pagano[12] is an extension of Reissner's theory for isotropic plates[14]. It is based upon the following assumed displacement field:

$$\begin{aligned} u(x,y,z) &= u^{o}(x,y) + z\psi_x(x,y) \\ v(x,y,z) &= v^{o}(x,y) + z\psi_y(x,y) \\ w(x,y,z) &= w^{o}(x,y) \end{aligned} \tag{1}$$

Here x,y are rectangular coordinates in the plane of the plate, z is the thickness-direction coordinate measured downward from the midplane of the plate; u,v,w are the displacements in the respective x,y,z directions; u^{o}, v^{o},w^{o} are the corresponding midplane displacements; and ψ_x and ψ_y are the slopes in the xz and yz planes due to bending only.

Neglecting body forces, body moments, and surface shearing forces, the equations of equilibrium can be written as

$$N_{1,x} + N_{6,y} = 0 \ ; \ N_{6,x} + N_{2,y} = 0$$

$$Q_{x,x} + Q_{y,y} + q = 0; \ M_{6,x} + M_{2,y} - Q_y = 0 \tag{2}$$

$$M_{1,x} + M_{6,y} - Q_x = 0$$

Here q is the normal pressure, $(\ )_{,x}$ denotes $\partial(\ )/\partial x$, and

$$(N_i, M_i) = \int_{-h/2}^{h/2} (1,z)\sigma_i \, dz \qquad (i=1,2,6)$$

$$(Q_y, Q_x) = \int_{-h/2}^{h/2} (\sigma_4, \sigma_5) \, dz \tag{3}$$

Here h is the plate (laminate) thickness, and the so-called contracted subscript notation is employed to denote the stress components. Thus, σ_1 and σ_2 are inplane normal stresses in the x and y directions; σ_6 is the inplane shear stress associated with the x,y axes; and σ_4 and σ_5 are the thickness shear stresses in the yz and xz planes.

Assuming that the only plane of symmetry existing is in the plane of the plate, the plate constitutive relations can be written as

$$\begin{Bmatrix} N_i \\ M_i \end{Bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} \\ B_{ij} & D_{ij} \end{bmatrix} \begin{Bmatrix} \varepsilon_j \\ \kappa_j \end{Bmatrix} \qquad (i,j=1,2,6)$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = \begin{bmatrix} K_4^2 S_{44} & K_4 K_5 S_{45} \\ K_4 K_5 S_{45} & K_5^2 S_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{Bmatrix} \tag{4}$$

The A_{ij}, B_{ij}, D_{ij}, S_{ij} are the respective inplane, bending-inplane coupling, bending or twisting, and thickness-shear stiffnesses defined as follows:

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} (1,z,z^2) Q_{ijk\ell} \, dz \qquad (i,j=1,2,6)$$

$$S_{ij} = \int_{-h/2}^{h/2} Q_{ijk\ell} \, dz \qquad (i,j=4,5) \tag{5}$$

Here $Q_{ijk\ell}$ denotes the plane-stress reduced stiffness, where ij refers to the position in the stress-strain-relation array (analogous to Eqs. (4)), k refers to the sign of the fiber-direction strain ($1 \sim +$, $2 \sim -$), and ℓ is the layer number.

The linear equations for the kinematics of deformation are

$$\begin{aligned} &\varepsilon_1^o = u,_x \quad ; \quad \varepsilon_2^o = v,_y \quad ; \quad \varepsilon_6^o = u,_y + v,_x \\ &\kappa_1 = \psi_{x,x} \; ; \quad \kappa_2 = \psi_{y,y} \; ; \quad \kappa_6 = \psi_{x,y} + \psi_{y,x} \\ &\varepsilon_4 = w,_y + \psi_y \quad ; \quad \varepsilon_5 = w,_x + \psi_x \end{aligned} \tag{6}$$

Equations (1)-(6) plus those of Appendix A constitute the appropriate theory, in differential-equation form, for the class of plates considered here (linear, thick, laminated, anisotropic, bimodulus).

Closed-Form Solution for Cross-Ply Laminate

Here we consider the particular case of a cross-ply laminate, i.e., one in which some of the layers are oriented along the x axis and the remainder along the y axis. Then the terms with subscripts 16, 26, and 45 vanish from the symmetric arrays in Eqs. (4). For bimodulus-material cross-ply laminates, Eqs. (5) integrate as indicated in Appendix A and depend upon the neutral-surface locations, Z_x and Z_y, as well as the $Q_{ijk\ell}$.

If it is tentatively assumed that the neutral-surface locations are independent of x and y, Eqs. (2), (4), and (6) can be combined to yield the following governing equations in terms of the midplane displacements (u^o,v^o,w^o) and bending slopes (ψ_y and ψ_x):

$$[L_{rs}]\{u^o,v^o,w^o,\psi_y,\psi_x\}^T = \{0,0,q,0,0\}^T \tag{7}$$

where $[L_{rs}]$ is a symmetric linear differential operator matrix with the following elements:

$$L_{11} = A_{11}d_x^2 + A_{66}d_y^2 \ ; \ L_{12} = (A_{12}+A_{66})d_xd_y \ ; \ L_{13} = 0 \ ;$$
$$L_{14} = (B_{12}+B_{66})d_xd_y \ ; \ L_{15} = B_{11}d_x^2 + B_{66}d_y^2 \ ; \ L_{22} = A_{66}d_x^2 + A_{22}d_y^2 \ ;$$
$$L_{23} = 0 \ ; \ L_{24} = B_{66}d_x^2 + B_{22}d_y^2 \ ; \ L_{25} = L_{14} \ ; \ L_{33} = - K_5^2S_{55}d_x^2 -$$
$$K_4^2 \ S_{44}d_y^2 \ ; \ L_{34} = - K_4^2S_{44}d_y \ ; \ L_{35} = - K_5^2 \ S_{55}d_x \ ; \tag{8}$$
$$L_{44} = D_{66}d_x^2 + D_{22}d_y^2 - K_4^2S_{44} \ ; \ L_{45} = (D_{12}+D_{66})d_xd_y \ ;$$
$$L_{55} = D_{11}d_x^2 + D_{66}d_y^2 - K_5^2S_{55} \ ; \ d_x = \partial(\ \ )/\partial x \ ; \ dy = \partial(\ \ )/\partial y$$

For a plate hinged flexurally, but free to move in a direction normal to each edge, the boundary conditions are

$$\begin{aligned}
N_1(0,y) = N_1(a,y) = 0 \ &; \ u^o(x,0) = u^o(x,b) = 0 \\
v^o(0,y) = v^o(a,y) = 0 \ &; \ N_2(x,0) = N_2(x,b) = 0 \\
w^o(0,y) = w^o(a,y) = 0 \ &; \ w^o(x,0) = w^o(x,b) = 0 \\
\psi_y(0,y) = \psi_y(a,y) = 0 \ &; \ M_2(x,0) = M_2(x,b) = 0 \\
M_1(0,y) = M_1(a,y) = 0 \ &; \ \psi_x(x,0) = \psi_x(x,b) = 0
\end{aligned} \tag{9}$$

The criteria that the neutral-surface locations associated with the x and y directions remain constant are as follows:

$$\varepsilon_1^o + z_{nx}\kappa_1 = 0 \ ; \ \varepsilon_2^o + z_{ny}\kappa_2 = 0 \tag{10}$$

or

$$\begin{aligned}
z_{nx} &= - \varepsilon_1^o/\kappa_1 = - u^o_{,x}/\psi_{x,x} \\
z_{ny} &= - \varepsilon_2^o/\kappa_2 = - v^o_{,y}/\psi_{y,y}
\end{aligned} \tag{11}$$

The normal-pressure loading is taken to be sinusoidally distributed as

$$q = q_o \ \sin \alpha x \sin \beta y \tag{12}$$

where $\alpha \equiv \pi/a$, $\beta \equiv \pi/b$.

The governing equations (7) with pressure distribution given by Eq. (12), boundary conditions (9), and neutral-surface location criteria (11) are all satisfied exactly in closed form by the following set of functions.

$$\begin{aligned} u^o &= U \cos \alpha x \sin \beta y \\ v^o &= V \sin \alpha x \cos \beta y \\ w^o &= W \sin \alpha x \sin \beta y \\ \psi_y &= Y \sin \alpha x \cos \beta y \\ \psi_x &= X \cos \alpha x \sin \beta y \end{aligned} \tag{13}$$

Then differential equation set (7) reduces to algebraic form as follows:

$$[C_{rs}]\{U,V,W,Y,X\}^T = \{0,0,q_o,0,0\}^T \tag{14}$$

Here $[C_{rs}]$ is a symmetric matrix with coefficients

$$\begin{aligned} &C_{11} = A_{11}\alpha^2 + A_{66}\beta^2 \ ; \ C_{12} = (A_{12}+A_{66})\alpha\beta \ ; \ C_{13} = 0 \ ; \\ &C_{14} = (B_{12}+B_{66})\alpha\beta \ ; \ C_{15} = B_{11}\alpha^2 + B_{66}\beta^2 \ ; \\ &C_{22} = A_{66}\alpha^2 + A_{22}\beta^2 \ ; \ C_{23} = 0 \ ; \ C_{24} = B_{66}\alpha^2 + B_{22}\beta^2 \ ; \\ &C_{25} = C_{14} \ ; \ C_{33} = K_5^2 S_{55}\alpha^2 + K_4^2 S_{44}\beta^2 \ ; \ C_{34} = K_4^2 S_{44}\beta \ ; \\ &C_{35} = K_5^2 S_{55}\alpha \ ; \ C_{44} = D_{66}\alpha^2 + D_{22}\beta^2 + K_4^2 S_{44} \ ; \\ &C_{45} = (D_{12}+D_{66})\alpha\beta \ ; \ C_{55} = D_{11}\alpha^2 + D_{66}\alpha^2 + K_5^2 S_{55} \end{aligned} \tag{15}$$

Finite-Element Analysis

Here we present a mixed finite-element model associated with Eqs. (1)-(4) governing the bending of laminated, thick composite plates. The word "mixed" implies that independent approximations are used for all of the variables, u, v, w, ψ_x, and ψ_y. Using the thin-plate equations of layered composite plates, and treating

$$w_{,x} + \psi_x = 0 \quad , \quad w_{,y} + \psi_y = 0 \tag{16}$$

as constraints, Reddy[17] presented a variational formulation of Eqs. (1)-(4). That is, the thick-plate theory can be interpreted as one resulting from the thin-plate theory by treating the slope-displacement relations as constraints. The Lagrange multipliers associated with these constraints are found to be the thickness-shear stress resultants, Q_x and Q_y. The model described here is essentially the same as in Ref. 17.

Suppose that the region occupied by the plate is given by $\Omega X(-h/2,h/2)$, where Ω denotes the middle plane (x-y). As noted earlier, the thickness direction is integrated into the coefficients, A_{ij}, B_{ij}, and D_{ij}. Hence, we divide the plate into a finite number of elements, denoted by Ω_e. Over each element Ω_e, we assume that the variables u, v, w, ψ_x, and ψ_y are interpolated by expressions of the form

$$\phi^e = \sum_i^n N_i \, \phi_e^i \tag{17}$$

where ϕ^e denotes the restriction of a typical variable to Ω_e, ϕ_e^i its value at node i (of element Ω_e), and N_i are the linearly independent interpolation functions associated with the typical element. Since we are concerned here with rectangular plates, the typical element is chosen to be the four-node (n=4) quadrilateral element of the serendipity family.

Substituting expressions of the form (17) for u,v,w,ψ_x, and ψ_y into the total potential energy associated with the case of a cross-ply laminate

$$\frac{1}{2}\int_{\Omega} \{A_{11}(\frac{\partial u}{\partial x})^2 + A_{66}(\frac{\partial u}{\partial y})^2 + 2A_{12}\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + 2A_{66}\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + A_{22}(\frac{\partial v}{\partial y})^2$$

$$+ A_{66}(\frac{\partial v}{\partial y})^2 + \frac{\partial u}{\partial x}(B_{11}\frac{\partial \psi_x}{\partial x} + B_{12}\frac{\partial \psi_y}{\partial y}) + B_{66}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x})$$

$$+ \frac{\partial v}{\partial y}(B_{12}\frac{\partial \psi_x}{\partial x} + B_{22}\frac{\partial \psi_y}{\partial y}) + \frac{\partial \psi_x}{\partial x}(B_{11}\frac{\partial u}{\partial x} + B_{12}\frac{\partial v}{\partial y}) + B_{66}(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x})(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$$

$$+ \frac{\partial \psi_y}{\partial y}(B_{12}\frac{\partial u}{\partial x} + B_{22}\frac{\partial v}{\partial y}) + D_{11}(\frac{\partial \psi_x}{\partial x})^2 + D_{66}(\frac{\partial \psi_x}{\partial y})^2$$

$$+ 2D_{12}\frac{\partial \psi_x}{\partial x}\frac{\partial \psi_y}{\partial y} + 2D_{66}\frac{\partial \psi_x}{\partial y}\frac{\partial \psi_y}{\partial x} + D_{22}(\frac{\partial \psi_y}{\partial y})^2 + D_{66}(\frac{\partial \psi_y}{\partial x})^2 + A_{55}(\frac{\partial w}{\partial x} + \psi_x)^2$$

$$+ A_{44}(\frac{\partial w}{\partial y} + \psi_y)^2\}\, dx\, dy + \int_{\Omega} qw\, dx\, dy \tag{18}$$

we obtain, for each element,

$$[K^e]\{\Delta^e\} = \{F^e\} \tag{19}$$

where $\{\Delta^e\} = \{u_i^e, v_i^e, w_i^e, \psi_{xi}^e, \psi_{yi}^e\}^T$, and

$$K_{ij}^{11} = A_{11}S_{ij}^{x} + A_{66}S_{ij}^{y},\quad K_{ij}^{12} = A_{12}S_{ij}^{xy} + A_{66}S_{ji}^{xy}$$

$$K_{ij}^{13} = 0,\quad K_{ij}^{14} = B_{11}S_{ij}^{x} + B_{66}S_{ij}^{y},\quad K_{ij}^{15} = B_{12}S_{ij}^{xy} + B_{66}S_{ji}^{xy}$$

$$K_{ij}^{22} = A_{22}S_{ij}^{y} + A_{66}S_{ij}^{x},\ K_{ij}^{23} = 0,\quad K_{ij}^{24} = B_{66}S_{ij}^{xy} + B_{12}S_{ji}^{xy}$$

$$K_{ij}^{25} = B_{66}S_{ij}^{x} + B_{22}S_{ij}^{y},\quad K_{ij}^{33} = A_{55}S_{ij}^{x} + A_{44}S_{ij}^{y},\ K_{ij}^{34} = A_{55}S_{ij}^{xo}$$

$$K_{ij}^{35} = A_{44}S_{ij}^{yo},\quad K_{ij}^{44} = D_{11}S_{ij}^{x} + D_{66}S_{ij}^{y} + A_{55}S_{ij}^{o}$$

$$K_{ij}^{45} = D_{12}S_{ij}^{xy} + D_{66}S_{ji}^{xy},\quad K_{ij}^{55} = D_{66}S_{ij}^{x} + D_{22}S_{ij}^{y} + A_{44}S_{ij}^{o}$$

$$F_i^3 = \int_{\Omega_e} qN_i\, dx\, dy,\ F_i^{\alpha} = 0,\ \alpha = 1,2,4,5,\quad A_{\alpha\beta} = K_{\alpha}K_{\beta}\, S_{\alpha\beta},\ (\alpha,\beta=4,5)$$

$$S_{ij}^{\xi\eta} = \int_{\Omega_e} N_{i,\xi}\, N_{j,\eta}\, dx\, dy\ (\xi,\eta=o,x,y),\quad S_{ij}^{o} = S_{ij}^{oo} \tag{20}$$

The element equations (19) are assembled in the usual manner, and boundary conditions are applied before solving the equations.

Numerical Results

As the first example, we take the case of a homogeneous (single-layer) plate of transversely isotropic bimodulus material. The plane of isotropy is assumed to coincide with the midplane of the plate, and the inplane Poisson's ratio is assumed to be zero. Then the closed-form solution reduces to the simplified form presented in Appendix B. Numerical results are presented in Tables 1 and 2.

Table 1. Comparison of Neutral-Surface Locations for Transversely Isotropic Square Plate

	Neutral-Surface Location Z		
$E_t/E_c=G_{zt}/G_{zc}$	$G_{zc}/E_c=0.1$	0.3	0.5
	Exact Closed-Form Solution:		
0.5	- 0.08578	- 0.08578	- 0.08578
1.0	0	0	0
2.0	+ 0.08578	+ 0.08578	+ 0.08578
	Simplified Approximate Solution:		
0.5	- 0.08579	- 0.08579	- 0.08579
1.0	0	0	0
2.0	+ 0.08579	+ 0.08579	+ 0.08579
	Mixed Finite-Element Solution:		
0.5	- 0.08578	- 0.08578	- 0.08578
1.0	0	0	0
2.0	+ 0.08578	+ 0.08578	+ 0.08578

Table 2. Comparison of Maximum Deflections for Transversely Isotropic Square Plate ($h/b=0.1$, $K^2 \equiv 5/6$)

	Dimensionless Deflection $WE_c h^3/q_o b^4$		
$E_t/E_c=G_{zt}/G_{zc}$	$G_{zc}/E_c=0.1$	0.3	0.5
	Exact Closed-Form Solution:		
0.5	0.05348	0.04774	0.04660
1.0	0.03688	0.03283	0.03201
2.0	0.02674	0.02387	0.02330
	Simplified Approximate Solution:		
0.5	0.05004	0.04660	0.04591
1.0	0.03445	0.03202	0.03153
2.0	0.02530	0.02342	0.02296
	Mixed Finite-Element Solution:		
0.5	0.05329	0.04743	0.04626
1.0	0.03675	0.03261	0.03178
2.0	0.02664	0.02371	0.02313

It is noted that the middle-surface location is independent of G_{zc} and G_{zt}. The agreement among the results obtained by all three solutions is quite good.

As examples of some actual bimodulus materials, aramid-cord/rubber and polyester-cord/rubber are selected. The material properties used are listed in Table 3. The data are based on test results of Patel et al.[3], using the data-reduction procedure of Model 2 in Ref. 4, except for the thickness-shear moduli, which were estimated as explained in Appendix C.

Table 3. Elastic Properties for Two Tire-Cord/Rubber, Unidirectional, Bimodulus Composite Materials[a]

Property and Units	Aramid/Rubber k=1	Aramid/Rubber k=2	Polyester Rubber k=1	Polyester Rubber k=2
Longitudinal Young's modulus, GPa	3.58	0.0120	0.617	0.0369
Transverse Young's modulus, GPa	0.00909	0.0120	0.00800	0.0106
Major Poisson's ratio, dimensionless[b]	0.416	0.205	0.475	0.185
Longitudinal-transverse shear modulus, GPa[c]	0.00370	0.00370	0.00262	0.00267
Transverse-thickness shear modulus, GPa	0.00290	0.00499	0.00233	0.00475

[a]Fiber-direction tension is denoted by k=1, and fiber-direction compression by k=2.
[b]It is assumed that the minor Poisson's ratio is given by the reciprocal relation.
[c]It is assumed that the longitudinal-thickness shear modulus is equal to this one.

Numerical results for single-layer rectangular plates with the fibers oriented parallel to the x axis are given in Table 4, while those for cross-ply plates (stacking sequence as described in Appendix A) are listed in Table 5.

As can be seen in Tables 4 and 5, the agreement between the closed-form and finite-element results for both neutral-surface position and deflection is extremely good. Thus, it can be considered that the finite-element analysis has been soundly validated, and can now be used for more complicated combinations of loading, geometry, and boundary conditions not amenable to closed-form solutions.

It is noted that the aramid-rubber plates, in both the single-ply and cross-ply cases, have noticeably larger values of Z_x than the polyester-rubber plates. This result is undoubtedly due to the more pronounced bimodulus effect in the fiber-direction Young's modulus of the aramid-rubber.

Table 4. Neutral-Surface Positions and Dimensionless Deflections for Rectangular Plates of Single-Layer 0° Aramid-Rubber and Polyester-Rubber, as Determined by Two Different Methods (Thickness/Width, h/b=0.1t; K^2=5/6)

Aspect Ratio	Z_x		Z_y		$WE_{22c}h^3/q_ob^4$	
	C.F.*	F.E.*	C.F.*	F.E.*	C.F.*	F.E.*
			Aramid-Rubber:			
0.5	0.4453	0.4454	- 0.3304	- 0.3007	0.002544	0.002750
0.6	0.4452	0.4452	- 0.2941	- 0.2734	0.004560	0.004827
0.7	0.4447	0.4447	- 0.2564	- 0.2419	0.007393	0.007712
0.8	0.4440	0.4440	- 0.2220	- 0.2117	0.01105	0.01140
0.9	0.4431	0.4431	- 0.1923	- 0.1846	0.01545	0.01582
1.0	0.4420	0.4420	- 0.1671	- 0.1614	0.02046	0.02083
1.2	0.4394	0.4394	- 0.1285	- 0.1250	0.03160	0.03193
1.4	0.4363	0.4363	- 0.1015	- 0.09919	0.04313	0.04335
1.6	0.4328	0.4329	- 0.08228	- 0.08070	0.05406	0.05416
1.8	0.4292	0.4294	- 0.06838	- 0.06724	0.06390	0.06388
2.0	0.4253	0.4254	- 0.05813	- 0.05727	0.07250	0.07236
			Polyester-Rubber:			
0.5	0.3044	0.3045	- 0.1597	- 0.1234	0.001529	0.001971
0.6	0.3044	0.3045	- 0.1538	- 0.1245	0.002652	0.003265
0.7	0.3042	0.3044	- 0.1426	- 0.1198	0.004283	0.005075
0.8	0.3039	0.3041	- 0.1299	- 0.1124	0.006517	0.007487
0.9	0.3035	0.3037	- 0.1174	- 0.1041	0.009421	0.01055
1.0	0.3029	0.3031	- 0.1061	- 0.09586	0.01303	0.01430
1.2	0.3015	0.3018	- 0.08728	- 0.08111	0.02223	0.02367
1.4	0.2999	0.3001	- 0.07329	- 0.06941	0.03348	0.03492
1.6	0.2979	0.2982	- 0.06296	- 0.06042	0.04574	0.04703
1.8	0.2957	0.2960	- 0.05528	- 0.05356	0.05793	0.05897
2.0	0.2934	0.2936	- 0.04959	- 0.04828	0.06925	0.07003

*C.F. denotes closed-form solution; F.E. signifies finite-element solution.

Table 5. Neutral-Surface Positions and Dimensionless Deflections for Rectangular Plates of Two-Layer Cross-Ply Aramid-Rubber and Polyester-Rubber, as Determined by Two Different Methods (Thickness/Width, h/b=0.1; K^2=5/6)

Aspect Ratio	Z_x		Z_y		$WE_{22c}h^3/q_ob^4$	
	C.F.*	F.E.*	C.F.*	F.E.*	C.F.*	F.E.*
			Aramid-Rubber:			
0.5	0.4433	0.4431	- 0.06343	- 0.06223	0.002472	0.002576
0.6	0.4427	0.4426	- 0.05478	- 0.05443	0.004388	0.004518
0.7	0.4418	0.4418	- 0.04794	- 0.04778	0.007072	0.007220
0.8	0.4407	0.4407	- 0.04247	- 0.04237	0.01054	0.01070
0.9	0.4396	0.4396	- 0.03803	- 0.03795	0.01475	0.01490
1.0	0.4384	0.4384	- 0.03437	- 0.03430	0.01957	0.01972
1.2	0.4356	0.4356	- 0.02883	- 0.02860	0.03043	0.03054
1.4	0.4326	0.4325	- 0.02470	- 0.02477	0.04185	0.04190
1.6	0.4292	0.4292	- 0.02160	- 0.02165	0.05282	0.05280
1.8	0.4257	0.4256	- 0.01922	- 0.01923	0.06277	0.06264
2.0	0.4219	0.4219	- 0.01735	- 0.01734	0.07151	0.07137
			Polyester-Rubber:			
0.5	0.3650	0.3652	- 0.1285	- 0.1256	0.002539	0.002732
0.6	0.3644	0.3646	- 0.1178	- 0.1164	0.004527	0.004772
0.7	0.3638	0.3639	- 0.1097	- 0.1089	0.007288	0.007575
0.8	0.3631	0.3631	- 0.1036	- 0.1031	0.01078	0.01109
0.9	0.3622	0.3622	- 0.09886	- 0.09859	0.01487	0.01519
1.0	0.3613	0.3613	- 0.09526	- 0.09502	0.01933	0.01966
1.2	0.3593	0.3593	- 0.09001	- 0.09000	0.02846	0.02879
1.4	0.3571	0.3570	- 0.08660	- 0.08660	0.03674	0.03707
1.6	0.3546	0.3545	- 0.08430	- 0.08430	0.04356	0.04389
1.8	0.3519	0.3518	- 0.08267	- 0.08267	0.04890	0.04925
2.0	0.3491	0.3490	- 0.08150	- 0.08150	0.05301	0.05337

*C.F. denotes closed-form solution; F.E. signifies finite-element solution.

Also, it is interesting to observe that there are only very slight differences in Z_x and deflection in going from a single layer to a cross-ply laminate. This is in contrast to the behavior of the polyester - rubber results and in considerable contrast to ordinary materials (which, of course have $Z_x = 0$ for the single-layer case). The most pronounced change in going from the single-layer to the two-layer case is the drastic decrease in Z_y for the aramid-rubber.

It should be cautioned that in the case of the closed-form solution, deflections due to various sinusoidally distributed loadings cannot be superimposed for bimodulus-material plates. The reason superposition is not valid here is that the necessary conditions for homogeneity of neutral-surface locations are not valid under superposition conditions, since, in general

$$Z_n = \frac{Z_n^{(1)} \; \psi_{x,x}^{(1)} (x,y) + Z_n^{(2)} \; \psi_{x,x}^{(2)} (x,y)}{\psi_{x,x}^{(1)} (x,y) + \psi_{x,x}^{(2)} (x,y)} \neq \text{constant}$$

even though $Z_n^{(1)}$, $Z_n^{(2)}$, ... for the various individual Fourier components are constants. However, the finite-element solution is not subject to these limitations, since it provides for stepwise variation in neutral-surface location. Fig. 1 shows results for both sinusoidally and uniformly distributed loadings.

Concluding Remarks

Both finite-element and closed-form solutions have been found for thick, rectangular plates of single-layer and cross-ply laminates of bimodulus materials. Excellent agreement was obtained, and thus the finite-element formulation of this problem is considered to have been validated against an accurate benchmark.

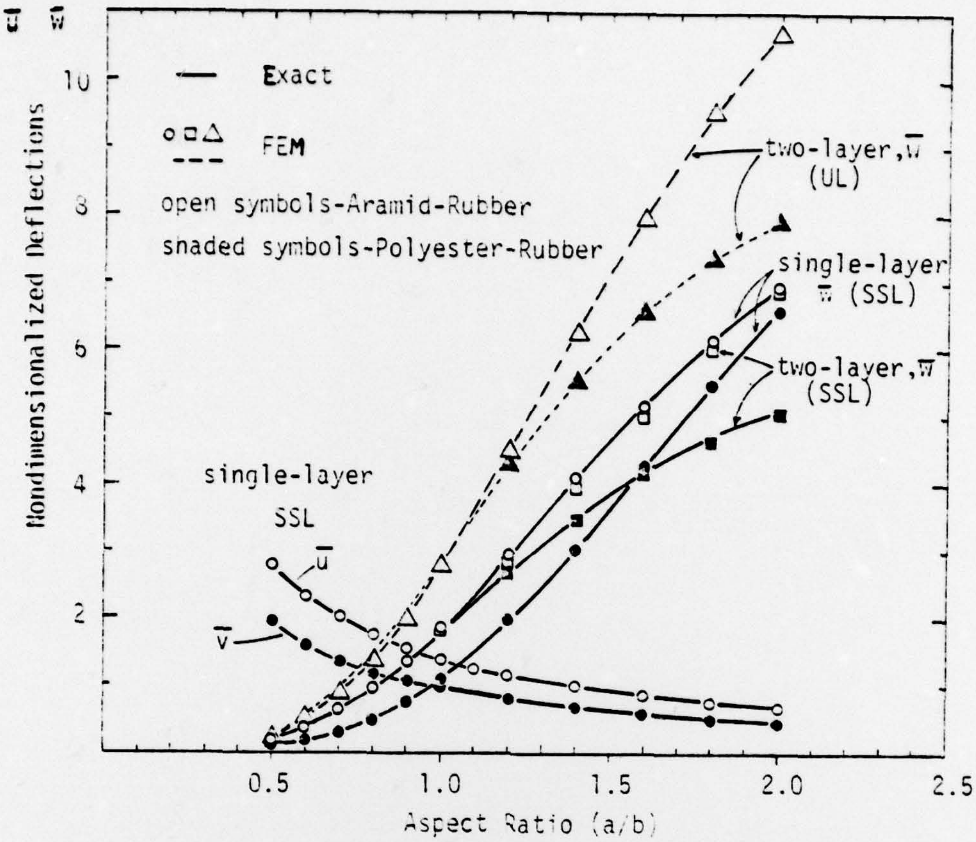

Fig. 1 Effect of plate aspect ratio on dimensionless displacements of aramid-rubber and polyester-rubber rectangular plates under uniform and sinusoidally distributed loadings (UL and SSL). Here $\bar{w}=wE_{22c}h^3/q_ob^4$ and $\bar{u}=10^3U/W$.

The research reported here is currently being extended to (1) thermal bending due to changes in midplane temperature and in gradient through the thickness, and (2) free vibration.

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Appendix A: Derivation of the Plate Stiffnesses for a Two-Layer Cross-Ply Laminate of Bimodulus Material

In laminates containing bimodulus materials, the results of evaluating the integrals for the plate stiffnesses, Eqs. (5), are more complicated than those for ordinary-material laminates, since the individual-layer plane-stress-reduced stiffnesses depend upon the neutral-surface location. Here we derive the expressions for the case of a two-layer, cross-ply laminate. Each layer has the same thickness and the same bimodulus orthotropic elastic properties with respect to the fiber direction. The bottom layer is denoted as layer 1, i.e., $\ell = 1$ in $Q_{ijk\ell}$, is oriented in the x direction, and occupies the thickness-direction interval from $z = 0$ to $z = h/2$, where z is measured position downward from the midplane. The top layer (ℓ=2) is oriented in the y direction and is located from $z = -h/2$ to $z = 0$. In the case considered, it is assumed that the upper portion of the top layer (ℓ=2) is in compression (k=2 in $Q_{ijk\ell}$) in the fiber direction and that the lower portion of the top layer is in tension (k=1), while the inner portion of the bottom layer (ℓ=1) is in compression (k=2) and the outer portion of this layer in tension (k=1). Summarizing, the four regions are as follows:

Layer	Region	Fiber-Direction Tension or Compression
$\ell = 2$	$-h/2$ to z_{ny}	Compression (k=2)
$\ell = 2$	z_{ny} to 0	Tension (k=1)
$\ell = 1$	0 to z_{nx}	Compression (k=2)
$\ell = 1$	z_{nx} to $h/2$	Tension (k=1)

It is noted that it is assumed that the x-direction neutral-surface location $z_{nx} \geq 0$, while the y-direction one (z_{ny}) is negative.

Thus, the integral for A_{ij} in Eqs. (5) is subdivided into four regions, in each of which the plane-stress reduced stiffnesses $Q_{ijk\ell}$ are constant.

$$A_{ij} = \int_{-h/2}^{z_{ny}} Q_{ij22}\, dz + \int_{z_{ny}}^{0} Q_{ij12}\, dz + \int_{0}^{z_{nx}} Q_{ij21}\, dz + \int_{z_{nx}}^{h/2} Q_{ij11}\, dz$$

$$= (Q_{ij11} + Q_{ij22})(h/2) + (Q_{ij21} - Q_{ij11})z_{nx} + (Q_{ij22} - Q_{ij12})z_{ny} \tag{A1}$$

Introducing $Z_x = z_{nx}/h$ and $Z_y = z_{ny}/h$, one obtains

$$A_{ij}/h = (1/2)(Q_{ij22} + Q_{ij11}) + (Q_{ij21} - Q_{ij11})Z_x + (Q_{ij22} - Q_{ij12})Z_y \tag{A2}$$

In similar fashion, the next two integrals in Eqs. (5) become

$$4B_{ij}/h^2 = (1/2)(Q_{ij11} - Q_{ij22}) + 2(Q_{ij21} - Q_{ij11})Z_x^2 + 2(Q_{ij22} - Q_{ij12})Z_y^2 \tag{A3}$$

$$12D_{ij}/h^3 = (1/2)(Q_{ij11} + Q_{ij22}) + 4(Q_{ij21} - Q_{ij11})Z_x^3 + 4(Q_{ij21} - Q_{ij12})Z_y^3 \tag{A4}$$

The expression for S_{ij}/h is the same as for A_{ij}/h, Eq. (A2).

To apply Eqs. (A2)-(A4) to a single-layer plate with the fibers oriented in the x direction, it is necessary to merely set $Z_y = 0$. In deriving these equations, it was assumed that $Z_x \geq 0$ and $Z_y \leq 0$. In the event that the final results obtained for Z_x and Z_y did not meet these conditions, obviously Eqs. (A2)-(A4) would not be valid, and it would become necessary to investigate another of the other three possible cases:

$Z_x \geq 0$ and $Z_y \geq 0$

$Z_x \leq 0$ and $Z_y \leq 0$

$Z_x \leq 0$ and $Z_y \geq 0$

Fortunately, however, in all of the static problems treated here, the conditions for the case derived in this appendix are satisfied.

Appendix B: Reduction of Closed-Form Solution for Single-Layer Transversely Isotropic Material

In order to obtain simple, concise expressions for the neutral-surface location and the deflection, the closed-form equations given in the body of the paper are reduced to the special case of a square plate made of a transversely isotropic bimodulus material with an inplane Poisson's ratio of zero. Then

$$\begin{aligned}
&A_{11} = A_{22} = A, \quad A_{12} = 0, \quad A_{66} = A/2 \\
&B_{11} = B_{22} = B, \quad B_{12} = 0, \quad B_{66} = B/2 \\
&D_{11} = D_{22} = D, \quad D_{12} = 0, \quad D_{66} = D/2 \\
&S_{44} = S_{55} = S, \quad S_{45} = 0
\end{aligned} \tag{B1}$$

Now Eqs. (15) reduce to the following, since $\beta = \alpha$:

$$\begin{aligned}
&C_{11} = (3/2)A\alpha^2, \quad C_{12} = (1/2)A\alpha^2, \quad C_{13} = 0 \\
&C_{14} = (1/2)B\alpha^2, \quad C_{15} = (3/2)B\alpha^2, \quad C_{ji} = C_{ij}, \\
&C_{22} = C_{11}, \quad C_{23} = 0, \quad C_{24} = C_{15}, \quad C_{25} = C_{14}, \\
&C_{33} = 2K^2S\alpha^2, \quad C_{34} = K^2S\alpha, \quad C_{35} = C_{34}, \\
&C_{44} = (3/2)D\alpha^2 + K^2S, \quad C_{45} = (1/2)D\alpha^2, \quad C_{55} = C_{44}
\end{aligned} \tag{B2}$$

The biaxial symmetry of this special case requires

$$b = a, \quad \beta = \alpha, \quad V = U, \quad Y = X, \quad z_{nx} = z_{ny} = z_n \tag{B3}$$

Using Eqs. (B2) and (B3) in the first two of Eqs. (14), one finds that for this special case

$$z_{nx} = z_{ny} = B/A = z_n \tag{B4}$$

Using the fourth equation of Eqs. (14), one obtains

$$X/W = \frac{-K^2 S\alpha}{2(D-Bz_n)\alpha^2 + K^2 S} \tag{B5}$$

It is noted that the bending slope vanishes for both $S = 0$ and $1/S = 0$.

Finally, the third of Eqs. (14) yields

$$W = \frac{q_o}{4(D-Bz_n)\alpha^4}\,[1 + 2(D-Bz_n)(\alpha^2/K^2 S)] \tag{B6}$$

It is seen that the quantity in front of the first bracket on the right side of Eq. (B6) is equal to the deflection of a thin isotropic plate. The second term inside the brackets is the fractional increase in deflection due to thickness-shear deformation, which obviously increases as G_z is decreased. The quantity $D-Bz_n$ is the so-called reduced stiffness, first obtained for laminated, isotropic thin plates by Pister[15].

For the present case, Eqs. (A2)-(A4) become

$$A/h = \bar{Q} + Z\Delta Q \quad , \quad 4B/h^2 = -(1/2)\Delta Q\,(1-4Z^2)$$
$$12D/h^3 = \bar{Q} + 4Z^3\Delta Q \tag{B7}$$

Here

$$\bar{Q} = (1/2)(Q_c+Q_t) \quad , \quad \Delta Q \equiv Q_c - Q_t \tag{B8}$$

Combining Eqs. (B4) and (B7), one obtains the following quadratic expression for Z:

$$Z = -(\bar{Q}/\Delta Q) \pm [(\bar{Q}/\Delta Q)^2 - (1/4)]^{1/2} \tag{B9}$$

Also

$$S/h = (1/2)(G_{zc}+G_{zt}) + Z(G_{zc}-G_{zt}) \tag{B10}$$

Appendix C: Method of Estimating Transverse-Thickness Shear Moduli

In the tests reported by Patel et al.[3], only inplane compliances were measured. Thus, it is necessary to estimate the values for the thickness-shear moduli, which are needed for the thick-plate analysis.

It is believed to be a reasonable engineering assumption to assume that an individual composite-material layer is transversely isotropic with the plane of isotropy being the cross-sectional plane, i.e., the plane normal to the fibers. Thus, it follows that the longitudinal-thickness shear modulus (G_{13}) is equal to the inplane (longitudinal-transverse) shear modulus (G_{12}).

Estimation of the other thickness-shear modulus is more complicated. One can use the well-known isotropic relation for the transverse-thickness shear modulus G_{23} in terms of the thickness Young's modulus E_3 and transverse-thickness Poisson's ratio ν_{23} provided the latter two quantities are known:

$$G_{23k} = E_{3k}/[2(1+\nu_{23k})] \tag{C1}$$

In view of the transverse-isotropy assumption mentioned above, it can be assumed that $E_{3k} = E_{2k}$, which was obtained from the inplane tests.

Foye[16] presented a relation for ν_{23} which can be rewritten in the following form, which is more convenient for the present purpose:

$$\nu_{23k} = \nu_{12k} + \frac{[(\nu_m/E_m) - (\nu_f/E_f)]\ \nu_m(1-\nu_m)}{\frac{1-\nu_m^2}{E_m(1-V_f)} + \frac{1}{E_f V_f} - \frac{\nu_m}{E_m}\frac{\nu_f}{E_f}} \tag{C2}$$

It is noted that Ref. 3 provided data for E_f, E_m, and V_f. It is reasonable to use a value of 0.499 for ν_m of rubber. Thus, the only unknown quantity on the right side of Eq. (C2) is ν_f, which could be computed from the following rule-of-mixtures expression for ν_{12k}, which is known to be very accurate for polymer-matrix composites:

$$\nu_{12k} = \nu_{fk}V_f + \nu_m(1-V_f) \tag{C3}$$

The rule-of-mixtures expression for the longitudinal Young's modulus is also known to be accurate for polymer-matrix composites:

$$E_{1k} = E_{fk}V_f + E_m(1-V_f) \tag{C4}$$

Unfortunately, however, for the data of Ref. 3, the measured values of E_{1t} (tension) were <u>higher</u> than predicted by Eq. (C4). Thus, it was decided to use Eq. (C4) to obtain an effective fiber volume fraction V_f^*, and then to use this effective value to predict E_{fc} (compression) and ν_{ft} and ν_{fc}. However, using either V_f or V_f^* in Eq. (C3), one obtains negative ν_{ft} and ν_{ft} values, which are not reasonable physically. Thus, it was assumed that due to the loose nature of the cord, that it was not restrained by the matrix. Thus, instead of obtaining ν_{fk} from Eq. (C3), it was obtained from

$$\nu_{fk} = \nu_{12k}/V_f^* \tag{C5}$$

Sample calculations for aramid-rubber in compression are as follows. From Eq. (C4) for k = t:

$$V_f^* = (E_{1t}-E_m)/(E_{ft}-E_m) = (3.58 - 0.0080)/(24.8 - 0.0080) = 0.144$$

Then, using Eq. (C4) for k = c:

$$E_{fc} = [E_{1c} - E_m(1-V_f^*)]/V_f^* = [0.00120 - 0.0080(0.856)]/0.144$$
$$= 0.0358 \text{ GPa}$$

From Eq. (C5) with k = c,

$$\nu_{fc} = \nu_{12c}/V_f^* = 0.205/0.144 = 1.42$$

Using Eq. (C2), one obtains $\nu_{23c} = 0.202$

Finally, from Eq. (C1), $G_{23c} = 0.00499$ GPa

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19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A mixed finite-element analysis is presented for static behavior of rectangular plates having finite transverse shear moduli and different elastic properties depending upon whether or not the fiber-direction strains are tensile or compressive. As a benchmark to evaluate the validity and accuracy of the finite-element analysis, a closed-form solution is presented for the particular case of an unsymmetric-cross-ply plate having freely supported edges and subjected to a sinusoidally distributed normal-pressure loading.

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